Providing a new graphical solution method of game with mixed strategies

Narjes Mashhadi Bandani[1]

Abstract: Currently, in two-person zero-sum games with randomized strategies we need to limit the size of either column or row player's strategies to 2 in order to use graphical method for determining the game value, i.e., a two-step procedure.
In this paper we expand the graphical method to include both the identification and solution of games in one step. Our method simplifies the procedure and enables us to solve medium size games efficiently.

Keywords: Games and Decisions, Mixed strategies, Graphical method, dominated

1.1 Introduction

For solving a game with mixed strategy there are many methods. On comparison, it is observed that the graphical method is much easier and quicker than other methods. For solving with graphical method we need to reduce matrix (each of columns or rows in to two) and then we can use the graphical method. But we know this reduction is not easy always, in some cases we need to calculate the average of two rows/columns and then we check the dominance. This consumes lot of time. Kumar (1999) provide another method for solving the game, but it seems that this method have 2 or 3 steps too, so in this paper has searched for a method to solve the game in just one step.

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1.2 Methodology

Suppose that you want to make decision on gaining player's (A) strategies. (Finding the min-max point) So draw columns instead of loosing player's (B) strategies. Then draw the strategies of player A. The distance of adjacent columns is unit.
Consider these points for reduction (min-max):
- Eliminate the strategies of player A which are dominated by other strategies.
- Eliminate the strategies of player B by comparing columns with each other. (By aid of player A strategies). Do not compare the column which has the maximum and minimum values.
- Check the previous points. It might be produced another new reduction.
- If the pay-off matrix do not been reduced still, get the average of 2 columns by drawing a vertical line just between 2 columns and compare it with other columns for reduction, there is must be a column for reduction.
- At last calculate the min-max value and strategies probabilities by graph.

To illustrate the procedure for solution, consider some examples.

1.3 Numerical Examples

1.3.1 Example-1

Pay-off matrix:

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} \begin{pmatrix} 2 & 7 & 2 \\ 3 & 5 & 4 \\ 11 & 2 & 12 \end{pmatrix} \end{array}$$

This is a two person zero-sum game, A is the gaining player and B is the loosing player. There is no saddle point.
We want to calculate Mini-max. Player B has three strategies so these should be considered as three vertical lines (the distance between lines is unit). Then we draw the player A strategies.

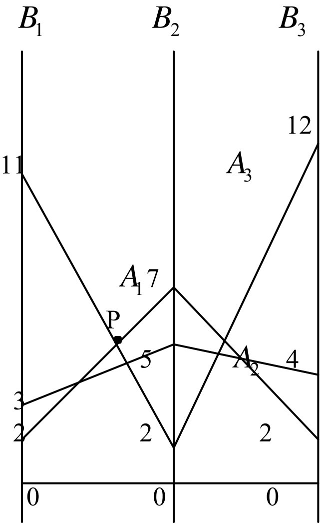


Fig. 1.3.1

As you see we can eliminate A_2 strategy. More over the ascending and descending slants between lines show us that we can't eliminate the B strategies by comparing the adjacent strategies. So we must compare the non-adjacent strategies. Evidently B_1 dominates B_3.

Clearly the maxi-min in the above diagram is P which determines the strategies A_1 and A_3 for the player A.

We can gain the value of game from diagram:

P-value = $\dfrac{(7\times 11)-(2\times 2)}{(7+11)-(2+2)}=\dfrac{73}{14}$

Because of B_3 and A_2 elimination:

$$p_2 = q_3 = 0$$

If the mini-max is closer to B_2 then player B choose this strategy mainly. So we have:

$$q_1 = \frac{5}{14} \quad , \quad q_2 = \frac{9}{14}$$

In addition we have this condition for player A, we get the following:

$$\frac{11-{}^{73}\!/_{14}}{11-2} = p_1 = \frac{9}{14} \quad , \quad p_3 = \frac{5}{14}$$

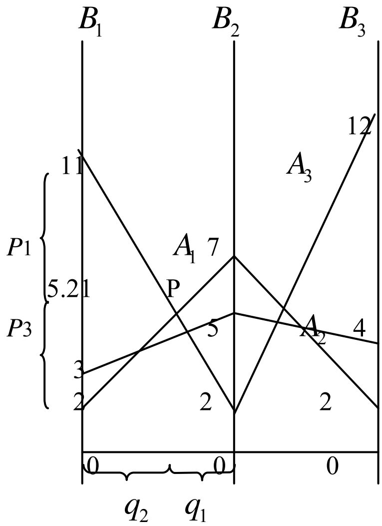


Fig. 1.3.2

1.3.2 Example-2

We have a pay-off matrix
We must calculate the Mini-max (or Maxi-min).

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{pmatrix} \end{array}$$

So we consider 4 vertical lines as player B strategies and draw the player A strategies.

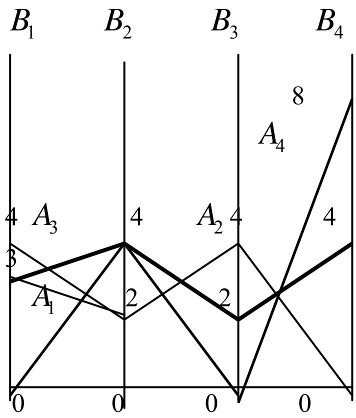


Fig. 1.3.3

We can eliminate A_1, ascending and descending slants between lines show us that we can't eliminate the B strategies by comparing the adjacent strategies. In addition B_4 has the highest and lowest value of strategies, so we can't compare it with other.

B_3 Dominates B_1, now we can calculate average of B_3 and B_4 (with drawing a virtual line between them) and after compare it with B_2, consequently B_2 is dominated.

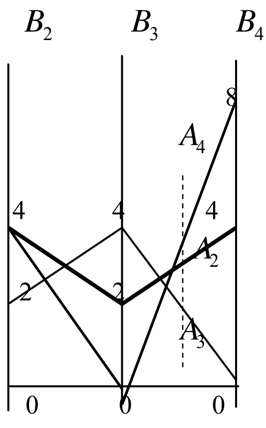


Fig. 1.3.4

Now we can eliminate A_2, we get

$$\text{P-value} = \frac{32-0}{12-0} = \frac{8}{3}$$

$$p_1 = p_2 = q_1 = q_2 = 0$$

If the mini-max is closer to B_3 then player B choose this strategy mainly. So we have:

$$q_3 = \frac{2}{3}, \; q_4 = \frac{1}{3}$$

In addition we have this condition for player A, we get the following

$$p_3 = \frac{2}{3} \;\; , \;\; p_4 = \frac{1}{3}$$

You can observe the probabilities as following.

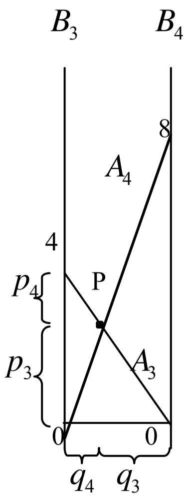


Fig. 1.3.5

1.3.3 Example-3

In this numerical example we have a comparison between previous method and present method.

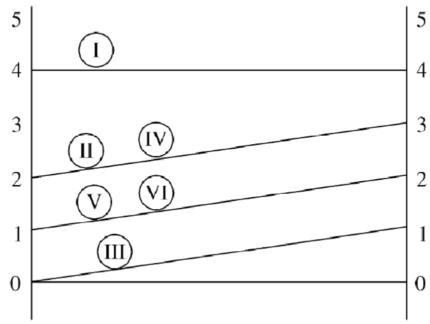
Previous method: Kumar (1999) has been solved this example in 3 steps:
There is a pay-off matrix

		Player B					
		I	II	III	IV	V	VI
Player A	1	4	2	0	2	1	1
	2	4	3	1	3	2	2
	3	4	3	7	−5	1	2
	4	4	3	4	−1	2	2
	5	4	3	3	−2	2	2

Arbitrarily selecting rows 1 and 2, matrix will be reduced to (2×6) matrix i.e. ,

	I	II	III	IV	V	VI
1	4	2	0	2	1	1
2	4	3	1	3	2	2

Solving by graphical method we get

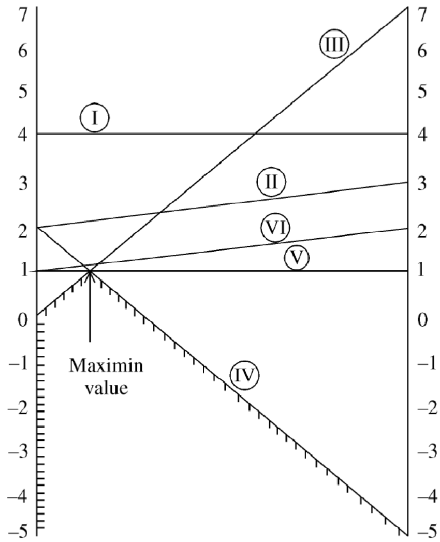


In the graphical solution, at least two lines should intersect to get the maxi-min or mini-max value. So we cannot select rows 1 and 2. We now shall go for next choice arbitrarily.

Let us select rows 1 and 3. Then the matrix is as follows.

	I	II	III	IV	V	VI
1	4	2	0	2	1	1
3	4	3	7	−5	1	2

Now solving by graphical method, we get



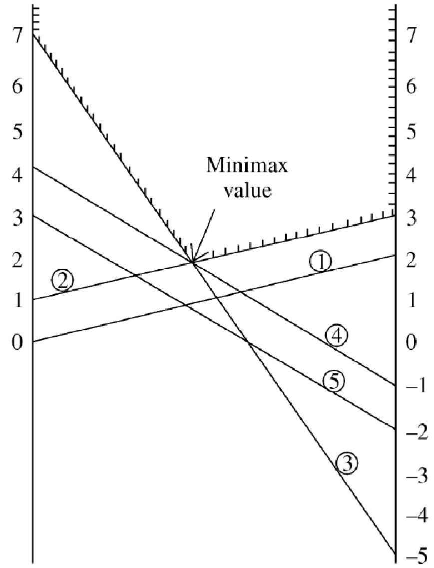
Now it is reduced to (2×2) matrix by selecting maxi-min value i.e.,

$$\begin{array}{c|cc|} & \text{III} & \text{IV} \\ \hline 1 & 0 & 2 \\ 3 & 7 & -5 \\ \hline \end{array}$$

Again, taking all the values of rows for those two particular columns i.e., III and IV, the matrix will be (5×2) matrix, i.e.,

$$\begin{array}{c|cc|} & \text{III} & \text{IV} \\ \hline 1 & 0 & 2 \\ 2 & 1 & 3 \\ 3 & 7 & -5 \\ 4 & 4 & -1 \\ 5 & 3 & -2 \\ \hline \end{array}$$

Again solving by graphical method, we get the following.



Now the reduced matrix is (2×2) matrix, i.e.,

	III	IV
2	1	3
3	7	−5

The required matrix is the reduced one of the original matrix. Therefore,

$$\text{value of game} = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$
$$= \frac{1 \times (-5) - (3 \times 7)}{(1 - 5) - (3 + 7)} = \frac{13}{7}.$$

Present method:

There is the same pay-off matrix

$$\begin{array}{cc} & \begin{array}{cccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ \text{Player A} \quad A_4 \\ A_5 \end{array} & \begin{pmatrix} 4 & 2 & 0 & 2 & 1 & 1 \\ 4 & 3 & 1 & 3 & 2 & 2 \\ 4 & 3 & 7 & -5 & 1 & 2 \\ 4 & 3 & 4 & -1 & 2 & 2 \\ 4 & 3 & 3 & -2 & 2 & 2 \end{pmatrix} \end{array}$$

We want to calculate maxi-min, so we draw 5 vertical lines as player A strategies

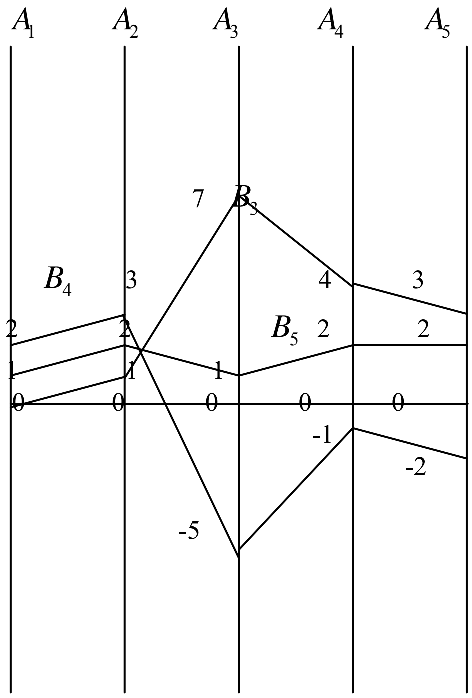


Fig. 1.3.6

There is no need for drawing B_1, B_2, B_6 strategies (since B_1, B_2, B_6 dominated by B_5)

In the graphical solution, at least two lines should intersect to get the maxi-min or mini-max value. So it is clear which point is P-value and remaining strategies are A_2, A_3, B_3, B_4

So we get the following easily

$$\text{P-value} = \frac{21+5}{10+4} = \frac{13}{7}$$

$$p_2 = \frac{6}{7} \quad , \quad p_3 = \frac{1}{7} \quad , \quad q_3 = \frac{3}{7} \quad , \quad q_4 = \frac{4}{7}$$

1.4 Limitation

It is to be noted that this paper highlights only in the context of Dominance property not in respect of other methods. Once the dominance property fails to get the solution, then the graphical method ($n \times m$ matrix) also fails.
For examples

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{pmatrix} 8 & 9 & 3 \\ 2 & 5 & 6 \\ 4 & 1 & 7 \end{pmatrix} \end{array}$$

The above matrix cannot reduce by dominance property. Hence the present methodology is not applicable to this problem.

1.5 Conclusion

To solve the game theory problems of two players by graphical method was restricted to ($2 \times n$) or ($m \times 2$) matrix only. If it is ($m \times n$) matrix, first we try to solve by dominance property. In the dominance property, we go on reducing row/column, step by step, till it reduce to (2×2) matrix. It is a lengthy procedure and more time consuming. This method is much quicker than the previous method (by Kumar (1999)). Previous method solve the problem in 2 or more steps, but this method solve the same problem just in one step (dominated steps and final value are obtained in one step).
So In this method, the same problem can be solved graphically. Using this method is so simple, quick and generic for everybody that is not much more familiar with game theory specially.

1.6 References

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