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A modified approach based on ranking fuzzy numbers for fuzzy integer programming with equality constraints

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Abstract This paper proposes a method for solving fuzzy integer programming problems where all the cost coefficients of the objective function and the right hand side terms of equality constraints are, in general, fuzzy numbers. We formulate a modified fuzzy ranking method to rank the fuzzy objective values and to deal with the equality relation on constraints under integrity conditions. We build a fuzzy subset in the integer decision space whose membership function represents the balance between the feasibility degree of constraints and the satisfaction degree of the goal. Finally, to illustrate our proposal, we solve a numerical example of a transport planning problem.

Keywords: Fuzzy integer programming, ranking fuzzy numbers, equality constraints, transport planning problem, uncertainty.

1.1 Introduction

Integer linear programming problems have an outstanding relevance in many fields, such as those related to production planning and transport planning problems when the product units are required to be defined with integer values. Furthermore, production and transport planning decisions are used to be made un-

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der uncertainty (Mula et al. 2006b). According to Mula et al (2006a), it can be distinguished between randomness or uncertainty corresponding to an objective variability in the model parameters, or epistemic uncertainty or lack of knowledge of the parameter values. Epistemic uncertainty, which is considered in this paper, is concerned with ill-known parameters modelled by fuzzy intervals in the setting of possibility theory (Zadeh 1978, Dubois and Prade 1988). Herrera and Verdegay (1995) present methods to solve fuzzy integer linear programming problems with either fuzzy constraints, or fuzzy numbers in the objective function or fuzzy numbers defining the set of constraints. These methods are based on the representation theorem and on fuzzy number ranking methods. However, the authors do not consider equality constraints.

This paper considers integer linear programming problems with equality constraints whose cost/profit coefficients of the objective function and right hand side terms of constraints are defined by fuzzy numbers but whose decision variables are crisp. In order to handle the relationship between the fuzzy left and the fuzzy right hand side of the constraints and to find the optimal value for the fuzzy objective function we propose a modified approach of the method of ranking fuzzy numbers by Jiménez (1996) and Jiménez et al. (2007) to solve integer linear programming problems. This method has been previously applied but for linear programming problems (Peidro et al. 2010). With the aim of validating our proposal, we apply it to a fuzzy integer transportation problem (FITP) with equality constraints. The parameters of each transportation problem are unit costs (profits) and demand and supply (production, storage capacity) values. In practice, these parameters are fuzzy in nature. Chanas and Kutcha (1998) propose an alternative algorithm to solve the transportation problem with crisp costs, fuzzy supply and demand values and the integrity condition imposed on the solution. The rest of the paper is structured as follows. Section 1.2 presents the FITP problem and the notation used. Section 1.3 develops the solution of the problem. Section 1.4 solves a FITP and compares the results of three methods: the proposal by Jiménez et al. (2007) for linear programming problems, dubbed as LFRN; the LFRN forcing the decision variables to be integer, dubbed as IFRN; and our proposal, dubbed as MFRN, which is the modification proposed in this paper to introduce the integer decision variables in LFRN to solve the unfeasibility problems that arise with IFRN. Finally, Section 1.5 provides the conclusions and further research.

1.2 Formulation of the problem and notation

The FITP considered in this paper can be described as follows. We assume a decision-maker who seeks to determine the right transportation planning of a homogeneous commodity from *m* sources to *n* destinations. Each destination is characterized by a forecasted demand which can be fulfilled with amounts of the commodity received from several sources, and each source has a total available supply capacity of the commodity to distribute to various destinations. The total available supply capacity for each source, the total forecast demand for each destination, and transport costs from each source to each destination are considered fuzzy due to incomplete or unobtainable information over the planning horizon. The purpose of the FITP is to minimize total transportation costs by using fully the available supply capacity at each source, and meeting the demand exactly at each destination. The sets of indices, parameters and decision variables for the FITP model are defined in the nomenclature (see Table 1).

Table 1 Nomenclature (fuzzy parameters are shown with a tilde: ~)

<table>
<tr><td colspan="2">Sets of indices</td><td colspan="2">Decision variables</td></tr>
<tr><td>i</td><td>Set of sources ($i=1,\ldots,I$)</td><td rowspan="2">X_{ij}</td><td rowspan="2">Units transported from source i to destination j (units)</td></tr>
<tr><td>j</td><td>Set of destinations ($j=1,\ldots,J$)</td></tr>
<tr><td colspan="4">Parameters</td></tr>
<tr><td>$\widetilde{c}_{ij}$</td><td>Transportation cost per unit delivered from source i to destination j (€/unit)</td><td rowspan="2">$\widetilde{D}_j$</td><td rowspan="2">Total forecast demand of each destination j (units)</td></tr>
<tr><td>$\widetilde{S}_i$</td><td>Total available supply for each source i (units)</td></tr>
</table>

The FITP is formulated as follows:

$$\text{Minimize } z \cong \sum_{i=1}^{I}\sum_{j=1}^{J}\widetilde{c}_{ij}X_{ij} \tag{1}$$

Subject to

$$\sum_{j=1}^{J} X_{ij} = \widetilde{S}_i \qquad \forall i \tag{2}$$

$$\sum_{i=1}^{I} X_{ij} = \widetilde{D}_j \qquad \forall j \tag{3}$$

$$X_{ij} \geq 0\text{, integer} \qquad \forall i\ \forall j \tag{4}$$

According to Liang (2008), in real-world transportation problems, constraints (2) and (3) are fuzzy in nature. Constraint (2) corresponds to the total available supply for each source i and constraint (3) is related to the total forecast demand for each destination j. The total available supply in constraint (2) for each source is commonly uncertain because available resources, worker skills, public policy and other factors are uncertain over the planning horizon. Additionally, the forecast demand in constraint (3) for each destination can never be determined precisely because the demand and supply in a dynamic market are uncertain. Moreover, transport costs are considered uncertain data and are modeled by fuzzy trapezoidal numbers $\widetilde{c}_{ij} = (c_{ij1}, c_{ij2}, c_{ij3}, c_{ij4})$, as well as, available supply $\widetilde{S}_i = (S_{i1}, S_{i2}, S_{i3}, S_{i4})$ and forecasted demand $\widetilde{D}_j = (D_{j1}, D_{j2}, D_{j3}, D_{j4})$.

1.3 Solution of the problem

1.3.1 Transformation of the fuzzy mixed-integer linear programming model into an equivalent crisp model according to Jiménez et al. (2007)

In this section, to address the fuzzy costs and right-hand side parameters of the FITP model, and to transform it into an equivalent auxiliary crisp integer linear programming model, we consider firstly the approach by Jiménez et al. (2007). Let us consider the following linear programming problem with fuzzy parameters:

$$\begin{aligned} &\text{Min } z = \tilde{c}^t x \\ &\text{s.a. } x \in N(\tilde{A},\tilde{b}) = \{x \in R^n \mid \tilde{a}_i x \geq \tilde{b}_i, i = 1,\dots,m, x \geq 0\} \end{aligned} \tag{5}$$

where $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_n)$, $\tilde{A} = [\tilde{a}_{ij}]_{mxn}$, $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n)^t$ represent, respectively, fuzzy parameters involved in the objective function and constraints. The possibility distribution of fuzzy parameters is assumed to be characterized by fuzzy numbers. $x = (x_1, x_2, \dots, x_n)$ is the crisp decision vector. We use a fuzzy relationship to compare fuzzy numbers that is computationally efficient to solve linear problems because it preserves its linearity (Jiménez, 1996). Thus, by applying the approach described by Jiménez et al. (2007) the problem (5) is transformed into the crisp equivalent parametric linear programming problem defined in (6).

$$\begin{aligned} &\text{Min } EV(\tilde{c})x \\ &\text{s.a. } [(1\text{-}\alpha)E_2^{a_i} + \alpha E_1^{a_i}]x \geq \alpha E_2^{b_i} + (1-\alpha)E_1^{b_i}, \quad i = 1,\dots,m, \quad x \geq 0, \quad \alpha \in [0,1] \end{aligned} \tag{6}$$

where α represents the degree that, at least, all the constraints are fulfilled; that is, α is the feasibility degree of a decision x. The expected value of a fuzzy number, noted $EV(\tilde{c})$, is the half point of its expected interval (Heilpern, 1992):

$$EV(\tilde{c}) = \frac{E_1^c + E_2^c}{2} \tag{7}$$

and if the fuzzy number $\tilde{c}$ is trapezoidal, its expected interval is easily calculated as follows:

$$EI(\tilde{c}) = [E_1^c, E_2^c] = [\frac{1}{2}(c_1 + c_2), \frac{1}{2}(c_3 + c_4)] \tag{8}$$

As (5) is considered an equality type constraint, this could be transformed into two equivalent crisp constraints:

$$\begin{aligned} &[(1\text{-}\frac{\alpha}{2})E_1^{a_i} + \frac{\alpha}{2}E_2^{a_i}]x \leq \frac{\alpha}{2}E_1^{b_i} + (1-\frac{\alpha}{2})E_2^{b_i}, \quad i = 1,\dots,m, \quad x \geq 0, \quad \alpha \in [0,1] \\ &[(1\text{-}\frac{\alpha}{2})E_2^{a_i} + \frac{\alpha}{2}E_1^{a_i}]x \geq \frac{\alpha}{2}E_2^{b_i} + (1-\frac{\alpha}{2})E_1^{b_i}, \quad i = 1,\dots,m, \quad x \geq 0, \quad \alpha \in [0,1] \end{aligned} \tag{9}$$

Consequently by applying this approach to the previously defined FITP model, and by considering trapezoidal fuzzy numbers for the uncertain parameters, we obtain an auxiliary crisp integer linear programming model as follows:

$$\text{Minimize } z = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{c_{ij1}+c_{ij2}+c_{ij3}+c_{ij4}}{4}\cdot X_{ij}\right) \tag{10}$$

Subject to

$$\sum_{j=1}^{J} X_{ij} \le \frac{\alpha}{2}\cdot\frac{S_{i1}+S_{i2}}{2}+\left(1-\frac{\alpha}{2}\right)\frac{S_{i3}+S_{i4}}{2} \qquad \forall i \tag{11}$$

$$\sum_{j=1}^{J} X_{ij} \ge \frac{\alpha}{2}\cdot\frac{S_{i3}+S_{i4}}{2}+\left(1-\frac{\alpha}{2}\right)\frac{S_{i1}+S_{i2}}{2} \qquad \forall i \tag{12}$$

$$\sum_{i=1}^{I} X_{ij} \le \frac{\alpha}{2}\cdot\frac{D_{j1}+D_{j2}}{2}+\left(1-\frac{\alpha}{2}\right)\frac{D_{j3}+D_{j4}}{2} \qquad \forall j \tag{13}$$

$$\sum_{i=1}^{I} X_{ij} \ge \frac{\alpha}{2}\cdot\frac{D_{j3}+D_{j4}}{2}+\left(1-\frac{\alpha}{2}\right)\frac{D_{j1}+D_{j2}}{2} \qquad \forall j \tag{14}$$

$$X_{ij} \ge 0,\ \text{integer},\ \alpha\in[0,1] \qquad \forall i \forall j \tag{15}$$

1.3.2 A modified approach based on fuzzy ranking numbers for fuzzy integer programming models with equality constraints

The previous approach is efficiently working for fuzzy linear programming problems (Peidro et al. 2010) but for fuzzy integer linear programming problems, where the integrity condition is imposed on the solution, there is a problem of unfeasibility of the solution. It happens because the right hand side of constraints (13) and (14) are equal fractional values, while the left hand side of these constraints, X_{ij}, must be integer values, what could be infeasible for certain values of α. To face with it, we propose to substitute the right hand side terms of constraints (13) and (14) for the corresponding most nearby integer values. Therefore, we have to add new auxiliary decision variables to ensure that the right hand side of constraints (13) and (14) can be transformed into integer and fractional values with the aim of getting the most nearby integer values. The model comprises of constraints (10)-(15) is modified as follows.

$$\text{Minimize } z = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{c_{ij1}+c_{ij2}+c_{ij3}+c_{ij4}}{4}\cdot X_{ij}\right)$$

$$+\sum_{i=1}^{I}\left(S1_i^{ABS}+S2_i^{ABS}\right)+\sum_{j=1}^{J}\left(D1_j^{ABS}+D2_j^{ABS}\right) \tag{16}$$

Subject to

$$S1_i^{INT} + S1_i^{DEC} = \frac{\alpha}{2} \cdot \frac{S_{i1} + S_{i2}}{2} + \left(1 - \frac{\alpha}{2}\right)\frac{S_{i3} + S_{i4}}{2} \qquad \forall i \tag{17}$$

$$S2_i^{INT} + S2_i^{DEC} = \frac{\alpha}{2} \cdot \frac{S_{i3} + S_{i4}}{2} + \left(1 - \frac{\alpha}{2}\right)\frac{S_{i1} + S_{i2}}{2} \qquad \forall i \tag{18}$$

$$\sum_{j=1}^{J} X_{ij} \le S1_i^{INT} \qquad \forall i \tag{19}$$

$$\sum_{j=1}^{J} X_{ij} \ge S2_i^{INT} \qquad \forall i \tag{20}$$

$$D1_j^{INT} + D1_j^{DEC} = \frac{\alpha}{2} \cdot \frac{D_{j1} + D_{j2}}{2} + \left(1 - \frac{\alpha}{2}\right)\frac{D_{j3} + D_{j4}}{2} \qquad \forall j \tag{21}$$

$$D2_j^{INT} + D2_j^{DEC} = \frac{\alpha}{2} \cdot \frac{D_{j3} + D_{j4}}{2} + \left(1 - \frac{\alpha}{2}\right)\frac{D_{j1} + D_{j2}}{2} \qquad \forall j \tag{22}$$

$$\sum_{i=1}^{I} X_{ij} \le D1_j^{INT} \qquad \forall j \tag{23}$$

$$\sum_{i=1}^{I} X_{ij} \ge D2_j^{INT} \qquad \forall j \tag{24}$$

$$S1_i^{DEC} \le S1_i^{ABS} \qquad \forall i \tag{25}$$

$$-S1_i^{DEC} \le S1_i^{ABS} \qquad \forall i \tag{26}$$

$$S2_i^{DEC} \le S2_i^{ABS} \qquad \forall i \tag{27}$$

$$-S2_i^{DEC} \le S2_i^{ABS} \qquad \forall i \tag{28}$$

$$D1_j^{DEC} \le D1_j^{ABS} \qquad \forall j \tag{29}$$

$$-D1_j^{DEC} \le D1_j^{ABS} \qquad \forall j \tag{30}$$

$$D2_j^{DEC} \le D2_j^{ABS} \qquad \forall j \tag{31}$$

$$-D2_j^{DEC} \le D2_j^{ABS} \qquad \forall j \tag{32}$$

$$S1_i^{INT}, S2_i^{INT} \ge 0, \text{ integer} \qquad \forall i \tag{33}$$

$$D1_j^{INT}, D2_j^{INT} \ge 0, \text{ integer} \qquad \forall j \tag{34}$$

$$S1_i^{ABS}, S2_i^{ABS} \le 0.999 \qquad \forall i \tag{35}$$

$$D1_j^{ABS}, D2_j^{ABS} \le 0.999 \qquad \forall j \tag{36}$$

$$S1_i^{ABS}, S2_i^{ABS} \ge 0 \qquad \forall i \tag{37}$$

$$D1_j^{ABS}, D2_j^{ABS} \ge 0 \qquad \forall j \tag{38}$$

$X_{ij} \geq 0$, integer, $\alpha \in [0,1]$ $\qquad \forall i \forall j \qquad$ (39)

Where the right-hand side coefficients of constraints (11) to (14) are represented by a sum of an integer variable and a real variable. For instance, the right-side hand coefficient of constraint (11) is equivalent to the sum of $S1_i^{INT}$ and $S1_i^{DEC}$. The same to constraint (12) and $S2_i^{INT}$ and $S2_i^{DEC}$ and consequently with constraints (13-14) and $D1_j^{INT}$, $D1_j^{DEC}$, $D2_j^{INT}$, $D2_j^{DEC}$. Then, right-hand side coefficients of constraints (11-14) are replaced by these integer variables in constraints (19-20) and (23-24). Hence, $S1_i^{DEC}$, $S2_i^{DEC}$, $D1_j^{DEC}$, $D2_j^{DEC}$ represent the deviation from original values in constraints (11-14) to integer values in constraints (19-20) and (23-24) and will be lower than 1. These deviations are expressed in a linear form of absolute value in constraints (25-28) and (29-32), respectively, by incorporating the variables $S1_i^{ABS}$, $S2_i^{ABS}$, $D1_j^{ABS}$, $D2_j^{ABS}$. Finally, the total sum of the absolute value of these deviations is added to the objective function to be minimized.

1.4 Numerical example

The proposed approach based on ranking fuzzy numbers will be illustrated at the following numerical example. We consider a network consisting of 3 sources and 3 destinations. Transport costs from sources to destinations, available supply and forecast demand are shown in Table 2, Table 3 and Table 4 as trapezoidal fuzzy numbers.

Table 2 Fuzzy transport costs from sources to destinations (in euros)

	Destination 1	Destination 2	Destination 3
Source 1	(1, 1.25, 1.5, 2)	(2, 2.5, 3, 3.25)	(3, 3.5, 3.75, 4.25)
Source 2	(2, 2.25, 2.75, 3)	(1, 1.25, 1.75, 2)	(2, 2.75, 3.5, 3.75)
Source 3	(3, 3.25, 3.75, 4)	(2, 2.25, 2.5, 2.75)	(1, 1.25, 1.75, 2)

Table 3 Fuzzy values of available supply at sources

	Available supply
Source 1	(2, 4, 6, 8)
Source 2	(3, 4, 7, 10)
Source 3	(4, 5, 7, 10)

Table 4 Fuzzy values of forecast demand at destinations

	Forescast demand
Destination 1	(5, 6, 9, 10)
Destination 2	(2, 4, 6, 7)
Destination 3	(2, 5, 8, 10)

The model has been implemented with the MPL 4.2 modelling language (2010). Resolution has been carried out with the optimisation solver Gurobi 4.6.1. Finally, a Microsoft Access 2010 database manages the input and output data of the model. Table 5 compares the expected values of total transportation costs, for several values of α, obtained from MFRN, LFRN and IFRN. The lowest expected values of total transportation costs are obtained by our modified approach, MRFN. The IFRN method obtains infeasible solutions for three values of α.

Table 5 Expected values of total transportation cost (in euros)

α	MRFN	LFRN	IFRN
0	17.69	19.69	21.69
0.1	17.69	20.62	22.13
0.2	18.75	21.55	22.13
0.3	21.69	22.48	26.13
0.4	21.69	23.41	26.13
0.5	22.13	24.34	26.13
0.6	23.19	25.27	28.69
0.7	23.19	26.32	29.56
0.8	26.13	27.51	infeasible
0.9	27.19	28.92	infeasible
1	27.19	27.66	infeasible

1.5 Conclusions

We have identified the unfeasibility of the solution for certain values of α, the feasibility degree of a decision x, when applying the approach based on ranking fuzzy numbers by Jiménez (1996) and Jiménez et al. (2007) for solving fuzzy integer linear programming problems with equality constraints. With the aim to cope with it, we have modified this approach, which has provided lower transportation costs for the FITP. A forthcoming work is applying this new modified approach for solving fuzzy goal programming models for material requirement planning under uncertainty and integrity conditions.

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